

Formulation of Electrodynamics with an External Source in the Presence of a Minimal Measurable Length

S. K. Moayed^{a*}, M. R. Setare^{b†}, B. Khosropour^{a‡}

^a *Department of Physics, Faculty of Sciences, Arak University, Arak 38156-8-8349, Iran*

^b *Department of Science, Campus of Bijar, University of Kurdistan Bijar, Iran*

Abstract

In a series of papers, Quesne and Tkachuk (J. Phys. A: Math. Gen. **39**, 10909 (2006); Czech. J. Phys. **56**, 1269 (2006)) presented a $D + 1$ -dimensional (β, β') -two-parameter Lorentz-covariant deformed algebra which leads to a nonzero minimal measurable length. In this paper, the Lagrangian formulation of electrodynamics in a $3+1$ -dimensional space-time described by Quesne-Tkachuk algebra is studied in the special case $\beta' = 2\beta$ up to first order over the deformation parameter β . It is demonstrated that at the classical level there is a similarity between electrodynamics in the presence of a minimal measurable length (generalized electrodynamics) and Lee-Wick electrodynamics. We obtain the free space solutions of the inhomogeneous Maxwell's equations in the presence of a minimal length. These solutions describe two vector particles (a massless vector particle and a massive vector particle). We estimate two different upper bounds on the isotropic minimal length. The first upper bound is near to the electroweak length scale ($\ell_{electroweak} \sim 10^{-18} m$), while the second one is near to the length scale for the strong interactions ($\ell_{strong} \sim 10^{-15} m$). The relationship between the Gaete-Spallucci nonlocal electrodynamics (J. Phys. A: Math. Theor. **45**, 065401 (2012)) and electrodynamics with a minimal length is investigated.

Keywords: Phenomenology of quantum gravity; Generalized uncertainty principle; Minimal length; Classical field theories; Classical electromagnetism; Quantum electrodynamics

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*E-mail: s-moayed@araku.ac.ir

†E-mail: rezakord@ipm.ir

‡E-mail: b-khosropour@phd.araku.ac.ir

1 Introduction

The unification between general theory of relativity and the Standard Model of particle physics is one of the most important problems in theoretical physics [1]. This unification predicts the existence of a minimal measurable length on the order of the Planck length. Also, recent studies in perturbative string theory and loop quantum gravity suggest that there is a minimal length scale in nature [2].

Today's we know that the existence of a minimal measurable length leads to an extended uncertainty principle. This extended uncertainty principle can be written as

$$\Delta X \Delta P \geq \frac{\hbar}{2} \left[1 + a_1 \left(\frac{l_P}{\hbar} \right)^2 (\Delta P)^2 + a_2 \left(\frac{l_P}{\hbar} \right)^4 (\Delta P)^4 + \dots \right], \quad (1)$$

where l_P is the Planck length and $a_i, \forall i \in \{1, 2, \dots\}$ are positive numerical constants [3,4]. If we keep only the first two terms on the right hand-side of (1), we will obtain the usual generalized uncertainty principle (GUP) as follows:

$$\Delta X \Delta P \geq \frac{\hbar}{2} \left[1 + a_1 \left(\frac{l_P}{\hbar} \right)^2 (\Delta P)^2 \right]. \quad (2)$$

It is obvious that in (2), ΔX is always greater than $(\Delta X)_{min} = \sqrt{a_1} l_P$. Many physicists believe that reformulation of quantum field theory in the presence of a minimal measurable length leads to a divergence free quantum field theory [5-7]. In the recent years, reformulation of quantum mechanics, gravity and quantum field theory in the presence of a minimal measurable length have been studied extensively [4-18]. The first attempt to construct the electromagnetic field in quantized space-time was made by H. S. Snyder [19]. In a previous work [14] we studied formulation of an electrostatic field with a charge density in the presence of a minimal length based on the Kempf algebra. In the present work we study formulation of electrodynamics with an external source in the presence of a minimal measurable length based on the Quesne-Tkachuk algebra. The organization of our paper is as follows. In Sec. 2, the $D+1$ -dimensional (β, β') -two-parameter Lorentz-covariant deformed algebra introduced by Quesne and Tkachuk is studied and it is shown that the Quesne-Tkachuk algebra leads to a minimal measurable length [20,21]. In Sec. 3, the Lagrangian formulation of electrodynamics with an external source in a $3+1$ -dimensional space-time described by Quesne-Tkachuk algebra is introduced in the special case $\beta' = 2\beta$, in which the position operators commute to first order in β . We show that at the classical level there is a similarity between electrodynamics in the presence of a minimal measurable length and Lee-Wick electrodynamics. In Sec. 4, the free space solutions of the inhomogeneous Maxwell's equations in the presence of a minimal measurable length are obtained. These solutions describe two different particles, a massless vector particle and a massive vector particle. In Sec. 5, we obtain two different upper bounds on the isotropic minimal length. One of these upper bounds on the isotropic minimal length is near to the

electroweak length scale ($\ell_{electroweak} \sim 10^{-18} m$). The second upper bound on the isotropic minimal length is near to the length scale for the strong interactions ($\ell_{strong} \sim 10^{-15} m$). In Sec. 6, we investigate the relation between electrodynamics in the presence of a minimal measurable length and the concept of nonlocality in electrodynamics. Our conclusions are presented in Sec. 7. SI units are used throughout this paper.

2 Lorentz-Covariant Deformed Algebra with a Minimal Observable Distance

Recently, Quesne and Tkachuk have introduced a Lorentz-covariant deformed algebra which describes a $D + 1$ -dimensional quantized space-time [20,21]. The Quesne-Tkachuk algebra in a $D + 1$ -dimensional space-time is specified by the following generalized commutation relations:

$$[X^\mu, P^\nu] = -i\hbar[(1 - \beta P_\rho P^\rho)g^{\mu\nu} - \beta' P^\mu P^\nu], \quad (3)$$

$$[X^\mu, X^\nu] = i\hbar \frac{2\beta - \beta' - (2\beta + \beta')\beta P_\rho P^\rho}{1 - \beta P_\rho P^\rho} (P^\mu X^\nu - P^\nu X^\mu), \quad (4)$$

$$[P^\mu, P^\nu] = 0, \quad (5)$$

where $\mu, \nu, \rho = 0, 1, 2, \dots, D$ and β, β' are two non-negative deformation parameters ($\beta, \beta' \geq 0$). In the above equations β and β' are constant parameters with dimension $(momentum)^{-2}$. Also, X^μ and P^μ are position and momentum operators in the deformed space and $g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1, -1, -1, \dots, -1)$. In the special case where $D = 3$ and $\beta = 0$, the Quesne-Tkachuk algebra (3)-(5) reduces to the Snyder algebra [22].

An immediate consequence of relation (3) is the appearance of an isotropic minimal length which is given by

$$(\Delta X^i)_0 = \hbar \sqrt{(D\beta + \beta')[1 - \beta \langle (P^0)^2 \rangle]} \quad , \quad \forall i \in \{1, 2, \dots, D\}. \quad (6)$$

In Ref. [23], Tkachuk introduced a representation which satisfies the generalized commutation relations (3)-(5) up to first order in deformation parameters β and β' .

The Tkachuk representation is given by

$$X^\mu = x^\mu - \frac{2\beta - \beta'}{4} (x^\mu p_\rho p^\rho + p_\rho p^\rho x^\mu), \quad (7)$$

$$P^\mu = (1 - \frac{\beta'}{2} p_\rho p^\rho) p^\mu, \quad (8)$$

where x^μ and $p^\mu = i\hbar \frac{\partial}{\partial x_\mu} = i\hbar \partial^\mu$ are position and momentum operators in ordinary relativistic quantum mechanics. In this study, we consider the special case $\beta' = 2\beta$, in which the position

operators commute to first order in deformation parameter β , i.e., $[X^\mu, X^\nu] = 0$. In this linear approximation, the Quesne-Tkachuk algebra becomes

$$[X^\mu, P^\nu] = -i\hbar[(1 - \beta P_\rho P^\rho)g^{\mu\nu} - 2\beta P^\mu P^\nu], \quad (9)$$

$$[X^\mu, X^\nu] = 0, \quad (10)$$

$$[P^\mu, P^\nu] = 0. \quad (11)$$

The following representations satisfy (9)-(11), in the first order in β :

$$X^\mu = x^\mu, \quad (12)$$

$$P^\mu = (1 - \beta p_\rho p^\rho)p^\mu. \quad (13)$$

Note that the representations (7), (8) and (12), (13) coincide when $\beta' = 2\beta$.

3 Lagrangian Formulation of Electrodynamics with an External Source in the Presence of a Minimal Length Based on the Quesne-Tkachuk Algebra

The Lagrangian density for a massless vector field $A^\mu = (\frac{1}{c}\phi, \mathbf{A})$ with an external source $J^\mu = (c\rho, \mathbf{J})$ in a 3 + 1-dimensional space-time is [24]

$$\mathcal{L} = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu, \quad (14)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor. In a 3 + 1-dimensional space-time the components of the electromagnetic field tensor $F_{\mu\nu}$ can be written as

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}. \quad (15)$$

The Euler-Lagrange equation for the vector field A^μ is

$$\frac{\partial \mathcal{L}}{\partial A_\lambda} - \partial_\rho \left(\frac{\partial \mathcal{L}}{\partial (\partial_\rho A_\lambda)} \right) = 0. \quad (16)$$

If we substitute the Lagrangian density (14) in the Euler-Lagrange equation (16), we will obtain the inhomogeneous Maxwell's equations as follows:

$$\partial_\rho F^{\rho\lambda} = \mu_0 J^\lambda. \quad (17)$$

The electromagnetic field tensor $F_{\mu\nu}$ satisfies the Bianchi identity

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0. \quad (18)$$

Equation (18) leads to the homogeneous Maxwell's equations. Now, we obtain the Lagrangian density for electrodynamics in the presence of a minimal observable distance based on the Quesne-Tkachuk algebra. For this purpose, let us write the Lagrangian density (14) by using the representations (12) and (13), i.e.,

$$x^\mu \longrightarrow X^\mu = x^\mu, \quad (19)$$

$$\partial^\mu \longrightarrow \nabla^\mu := (1 + \beta \hbar^2 \square) \partial^\mu, \quad (20)$$

where $\square := \partial_\mu \partial^\mu$ is the d'Alembertian operator. The result reads

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4\mu_0} (\nabla_\mu A_\nu - \nabla_\nu A_\mu) (\nabla^\mu A^\nu - \nabla^\nu A^\mu) - J^\mu A_\mu \\ &= -\frac{1}{4\mu_0} [(1 + \beta \hbar^2 \square) \partial_\mu A_\nu - (1 + \beta \hbar^2 \square) \partial_\nu A_\mu] \\ &\quad [(1 + \beta \hbar^2 \square) \partial^\mu A^\nu - (1 + \beta \hbar^2 \square) \partial^\nu A^\mu] - J^\mu A_\mu \\ &= -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4\mu_0} (\hbar \sqrt{2\beta})^2 F_{\mu\nu} \square F^{\mu\nu} \\ &\quad - J^\mu A_\mu + \mathcal{O}((\hbar \sqrt{2\beta})^4). \end{aligned} \quad (21)$$

The term $-\frac{1}{4\mu_0} (\hbar \sqrt{2\beta})^2 F_{\mu\nu} \square F^{\mu\nu}$ in the above Lagrangian can be considered as a minimal length effect.

If we neglect terms of order $(\hbar \sqrt{2\beta})^4$ and higher in (21), we will obtain the following Lagrangian density

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4\mu_0} (\hbar \sqrt{2\beta})^2 F_{\mu\nu} \square F^{\mu\nu} - J^\mu A_\mu. \quad (22)$$

The Lagrangian density (22) is similar to the Abelian Lee-Wick model which was introduced by Lee and Wick as a finite theory of quantum electrodynamics [25-29]. The equation (22) can be written as follows:

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} + \frac{1}{4\mu_0} (\hbar \sqrt{2\beta})^2 (\partial_\alpha F_{\mu\nu}) (\partial^\alpha F^{\mu\nu}) + \partial_\alpha \chi^\alpha - J^\mu A_\mu, \quad (23)$$

where

$$\chi^\alpha := -\frac{1}{4\mu_0} (\hbar \sqrt{2\beta})^2 F_{\mu\nu} \partial^\alpha F^{\mu\nu}. \quad (24)$$

After dropping the total derivative term $\partial_\alpha \chi^\alpha$, the Lagrangian density (23) will be equivalent to the following Lagrangian density:

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} + \frac{1}{4\mu_0} (\hbar\sqrt{2\beta})^2 (\partial_\alpha F_{\mu\nu})(\partial^\alpha F^{\mu\nu}) - J^\mu A_\mu. \quad (25)$$

Using the Bianchi identity (18) and dropping the total derivative terms, the expression (25) can also be written as follows:

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2\mu_0} a^2 (\partial_\sigma F^{\rho\sigma})(\partial^\beta F_{\rho\beta}) - J^\mu A_\mu, \quad (26)$$

where $a := \hbar\sqrt{2\beta}$. Equation (26) is the Lagrangian density originally introduced by Podolsky [30-33], and a is called Podolsky's characteristic length [34-38]. The Euler-Lagrange equation for the Lagrangian density (25) is [39,40]

$$\frac{\partial \mathcal{L}}{\partial A_\lambda} - \partial_\rho \left(\frac{\partial \mathcal{L}}{\partial (\partial_\rho A_\lambda)} \right) + \partial_\sigma \partial_\rho \left(\frac{\partial \mathcal{L}}{\partial (\partial_\sigma \partial_\rho A_\lambda)} \right) = 0. \quad (27)$$

If we substitute (25) into (27), we will obtain the inhomogeneous Maxwell's equations in the presence of a minimal observable distance as follows:

$$\partial_\rho F^{\rho\lambda} + (\hbar\sqrt{2\beta})^2 \square \partial_\rho F^{\rho\lambda} = \mu_0 J^\lambda. \quad (28)$$

It should be mentioned that equations (28) have been previously obtained from a different perspective by Kober [41]. Equations (18) and (28) can be written in the vector form as follows:

$$\nabla \cdot \mathbf{E} + (\hbar\sqrt{2\beta})^2 \square (\nabla \cdot \mathbf{E}) = \frac{\rho}{\varepsilon_0}, \quad (29)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (30)$$

$$\nabla \times \mathbf{B} + (\hbar\sqrt{2\beta})^2 \square (\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}) = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}, \quad (31)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (32)$$

The generalized Maxwell's equations (29)-(32) have been introduced earlier by Tkachuk in Ref. [23] with a different approach. In the limit $\hbar\sqrt{2\beta} \rightarrow 0$, the generalized inhomogeneous Maxwell's equations (29) and (31) become the usual inhomogeneous Maxwell's equations.

4 Free Space Solutions of the Generalized Inhomogeneous Maxwell's Equations

In this section, we obtain the plane wave solutions of the generalized inhomogeneous Maxwell's equations (28) in a 3 + 1-dimensional space-time.

In free space ($\rho = 0, \mathbf{J} = 0$), equations (28) can be written as

$$\partial_\rho F^{\rho\lambda} + (\hbar\sqrt{2\beta})^2 \square \partial_\rho F^{\rho\lambda} = 0. \quad (33)$$

In the Lorentz gauge ($\partial_\rho A^\rho = 0$) the field equations (33) become

$$\square A^\lambda + (\hbar\sqrt{2\beta})^2 \square \square A^\lambda = 0. \quad (34)$$

Now, we try to find a plane wave solution of (34):

$$A^\lambda(x) = A e^{-\frac{i}{\hbar} p \cdot x} \epsilon^\lambda(p), \quad (35)$$

where $\epsilon^\lambda(p)$ is the polarization four-vector and A is a normalization constant. In the above equation $p^\mu = (\frac{E}{c}, \mathbf{p})$ is the momentum four-vector. If we substitute (35) in (34), we will obtain

$$p^2(1 - \frac{a^2}{\hbar^2} p^2) = 0, \quad (36)$$

where $p^2 = p_\mu p^\mu = (\frac{E}{c})^2 - \mathbf{p}^2$.

Equation (36) leads to the following energy-momentum relations

$$E^2 = c^2 \mathbf{p}^2, \quad (37)$$

$$E^2 = m_{eff}^2 c^4 + c^2 \mathbf{p}^2, \quad (38)$$

where

$$m_{eff} := \frac{\hbar}{ac}. \quad (39)$$

Equation (37) describes a massless vector particle whereas equation (38) describes a massive vector particle with the effective mass m_{eff} .

5 Upper Bound Estimation of the Minimal Length in Generalized Electrodynamics

Substituting $\beta' = 2\beta$ into (6), and remembering $a = \hbar\sqrt{2\beta}$, we have

$$(\Delta X^i)_0 = \sqrt{\left(\frac{D+2}{2}\right) a^2 [1 + \mathcal{O}(a^2)]} \quad , \quad \forall i \in \{1, 2, \dots, D\}. \quad (40)$$

If we neglect terms of order a^4 and higher in (40), the isotropic minimal length in a $3 + 1$ -dimensional space-time becomes

$$(\Delta X^i)_0 \simeq \frac{\sqrt{10}}{2} a \quad , \quad \forall i \in \{1, 2, 3\}. \quad (41)$$

Now we are ready to estimate the upper bounds on the isotropic minimal length in generalized electrodynamics.

5.1 Upper Bound on the Isotropic Minimal Length Based on the Anomalous Magnetic Moment of the Electron

In a series of papers, Accioly and co-workers [27,29,34] have estimated an upper bound on Podolsky's characteristic length a by computing the anomalous magnetic moment of the electron in the framework of Podolsky's electrodynamics. This upper bound on a is [27,29,34]

$$a \leq 4.7 \times 10^{-18} \text{ m}. \quad (42)$$

Inserting equation (42) into equations (39) and (41), we find

$$m_{eff} \geq 41.8 \frac{\text{GeV}}{c^2} \text{ ,} \quad (43)$$

$$(\Delta X^i)_0 \leq 7.4 \times 10^{-18} \text{ m}. \quad (44)$$

5.2 Upper Bound on the Isotropic Minimal Length Based on the Ground State Energy of the Hydrogen Atom

In Ref. [37], Cuzinatto and co-workers have studied the influence of Podolsky's electrostatic potential on the ground state energy of the hydrogen atom. In their study, the upper limit on a is

$$a \leq 5.56 \times 10^{-15} \text{ m}. \quad (45)$$

Inserting equation (45) into equations (39) and (41), we find

$$m_{eff} \geq 35.51 \frac{\text{MeV}}{c^2} \text{ ,} \quad (46)$$

$$(\Delta X^i)_0 \leq 8.79 \times 10^{-15} \text{ m}. \quad (47)$$

It should be noted that the upper bound (47) is about three orders of magnitude larger than the upper bound (44), i.e.,

$$(\Delta X^i)_0 \text{ Ground State Energy of the Hydrogen Atom} \sim 10^3 (\Delta X^i)_0 \text{ Anomalous Magnetic Moment of the Electron} \text{ ,} \quad (48)$$

while the lower bound (46) is about three orders of magnitude smaller than the lower bound (43), i.e.,

$$m_{eff- \text{ Ground State Energy of the Hydrogen Atom}} \sim 10^{-3} m_{eff- \text{ Anomalous Magnetic Moment of the Electron}} . \quad (49)$$

6 Relationship between Nonlocal Electrodynamics and Electrodynamics in the Presence of a Minimal Length

In a series of papers, Smailagic and Spallucci [42-44] have introduced an approach to formulate non-commutative quantum field theory. Using Smailagic-Spallucci approach, Gaete and Spallucci introduced a nonlocal Lagrangian density for the vector field A^μ with an external source J^μ as follows:

$$\mathcal{L} = - \frac{1}{4\mu_0} F_{\mu\nu} \exp(\theta \square) F^{\mu\nu} - J^\mu A_\mu , \quad (50)$$

where θ is a constant parameter with dimensions of $(length)^2$ [45]. We assume that the function $\exp(\theta \square)$ in (50) can be expanded in a power series as follows:

$$\exp(\theta \square) = \sum_{l=0}^{\infty} \frac{\theta^l}{l!} \square^l , \quad (51)$$

where \square^l denotes the \square operator applied l times [46].

If we insert (51) into (50), we will obtain the following Lagrangian density

$$\mathcal{L} = - \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4\mu_0} \theta F_{\mu\nu} \square F^{\mu\nu} - \frac{1}{4\mu_0} \sum_{l=2}^{\infty} \frac{\theta^l}{l!} F_{\mu\nu} \square^l F^{\mu\nu} - J^\mu A_\mu . \quad (52)$$

After neglecting terms of order θ^2 and higher in (52) we obtain

$$\mathcal{L} = - \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4\mu_0} \theta F_{\mu\nu} \square F^{\mu\nu} - J^\mu A_\mu . \quad (53)$$

A comparison between equations (22) and (53) shows that there is an equivalence between the Gaete-Spallucci electrodynamics to first order in θ and the Lee-Wick electrodynamics (or electrodynamics in the presence of a minimal length). The relationship between the non-commutative parameter θ in (53) and $a = \hbar\sqrt{2\beta}$ in (22) is

$$a = \sqrt{\theta} . \quad (54)$$

Inserting equation (54) into equations (39) and (41), we find

$$m_{eff} = \frac{\hbar}{\sqrt{\theta} c} , \quad (55)$$

$$(\Delta X^i)_0 \simeq \frac{\sqrt{10} \theta}{2} \quad , \quad \forall i \in \{1, 2, 3\}. \quad (56)$$

Using (45) in (54), we obtain the following upper bound for the non-commutative parameter θ :

$$\theta_{\text{Ground State Energy of the Hydrogen Atom}} \leq 3.09 \times 10^{-29} \text{ m}^2. \quad (57)$$

The above upper bound on the non-commutative parameter θ , i.e., $3.09 \times 10^{-29} \text{ m}^2$ is near to the neutron-proton scattering cross section (10^{-25} cm^2) [47]. It is necessary to note that the electrodynamics in the presence of a minimal observable distance is only correct to the first order in the deformation parameter β , while the Gaete-Spallucci electrodynamics is valid to all orders in the non-commutative parameter θ .

7 Conclusions

Heisenberg believed that every theory of elementary particles should contain a minimal observable distance of the magnitude $\ell_0 \sim 10^{-13} \text{ cm}$ [47-50]. He hoped that the introduction of a minimal length would eliminate divergences that appear in quantum electrodynamics. Today's we know that every theory of quantum gravity predicts the existence of a minimal measurable length which leads to a GUP. An immediate consequence of the GUP is a generalization of position and derivative operators according to equations (19) and (20) for $\beta' = 2\beta$. It was shown that the Lagrangian formulation of electrodynamics with an external source in the presence of a minimal measurable length leads to the inhomogeneous fourth-order field equations. We demonstrated the similarity between electrodynamics in the presence of a minimal length and Lee-Wick electrodynamics. We have shown that the free space solutions of the inhomogeneous Maxwell's equations in the presence of a minimal length describe two particles, a massless vector particle and a massive vector particle with the effective mass $m_{eff} = \frac{\hbar}{ac}$. Now, let us compare the upper bounds on the isotropic minimal length in this paper with the results of Refs. [47-51]. The upper limit on the isotropic minimal length in equation (44) is near to the electroweak length scale ($\ell_{electroweak} \sim 10^{-18} \text{ m}$) [51], while the upper limit (47) is near to the minimal observable distance which was proposed by Heisenberg ($\ell_0 \sim 10^{-13} \text{ cm}$) [47-50]. It is interesting to note that the lower bound on the effective mass m_{eff} in equation (43), i.e., $41.8 \frac{\text{GeV}}{c^2}$ is of the same order of magnitude as the mass of the W^\pm and Z^0 vector bosons ($M_W = 80.425 \pm 0.038 \frac{\text{GeV}}{c^2}$, $M_Z = 91.1876 \pm 0.0021 \frac{\text{GeV}}{c^2}$) [52]. Finally, we have investigated the relationship between the Gaete-Spallucci nonlocal electrodynamics and electrodynamics with a minimal length.

Note added

After this work was completed, we became aware of an interesting article by Maziashvili and Megrelidze [53], in which the authors study the electromagnetic field in the presence of a momentum cutoff. For their discussion they use the following modified Heisenberg algebra

$$[X^i, P^j] = i\hbar \left(\frac{2\beta \mathbf{P}^2}{\sqrt{1 + 4\beta \mathbf{P}^2} - 1} \delta^{ij} + 2\beta P^i P^j \right), \quad (58)$$

$$[X^i, X^j] = 0, \quad (59)$$

$$[P^i, P^j] = 0, \quad (60)$$

where $i, j = 1, 2, 3$ and β is a deformation parameter [54]. In our work we have formulated electrodynamics in the framework of Quesne-Tkachuk algebra which is a Lorentz-covariant deformed algebra whereas the authors of Ref. [53] have studied electrodynamics in the framework of (58)-(60) algebra which is not a Lorentz-covariant algebra.

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